2. Derivation of IPCC expression $\Delta F = 5.35 \, ln \, (C/C_0)$

2.1 Derivation One

The assumptions we will make allow us to represent the real atmosphere. This remarkably reasonable representation of the real atmosphere is due in part to the small mean optical thickness of the Earth's atmosphere. We assume that the atmosphere is transparent to visible radiation and heating only occurs at the Earth's Finally, we assume local thermodynamic equilibrium. This means that in a localised atmospheric volume below 40kms we consider it to be isotropic (emission is non-directional) with a uniform temperature. Here emissivity equals absorptivity. Two temperatures (T_e and T_s) are important. The *effective emission temperature* (T_e) is the temperature the Earth would have without an atmosphere just taking into account its reflectivity and its distance from the sun. The Earth radiates as a black body in the Infrared spectrum. We calculate the effective emission temperature by assuming the rate of the Earth's energy absorption equals the rate of emission.

$$S\pi r^2(1 - \alpha_p) = 4\pi r^2 \sigma T_e^4$$

Where the solar constant, S = 1366 W/m^2 and the planetary albedo, α_p = 0.3244

Stefan-Boltzmann law for the Earth as a black body (or perfect radiator) gives:

$F = \sigma T^4$

where F is the flux density emitted in W/m^2
σ is the Stefan-Boltzmann constant, and
T is the absolute temperature.

$$F = \sigma T_e^4 = F_e$$

Therefore, the flux (F) absorbed by the climate system is:

$$F_e = S\frac{(1-\alpha_p)}{4} \qquad (4)$$

T_s is the surface air temperature and $F_{g \blacktriangleright a}$ (**g**round to **a**tmosphere) is the upward flux density (heat) radiated from the surface (σT_s^4).

First we calculate the vertical opacity of the atmosphere (τ_g) from the Chamberlain[4] expression that he derived from the general heat transfer equation:

$$T_s^4 = T_e^4\left(1 + \frac{3}{4}\tau_g\right) \qquad (5)$$

$$T_s^4 = \frac{F_e}{\sigma}\left(1 + \frac{3}{4}\tau_g\right) = \frac{(1 - \alpha_p)S}{4\sigma}\left(1 + \frac{3}{4}\tau_g\right) \; from\ Equations\ 4\ and\ 5$$

$$F = \sigma T_s^4 = \frac{(1-\alpha_p)S}{4}\left(1 + \frac{3}{4}\tau_g\right) = F_{g\blacktriangleright a} \quad (6)$$

We differentiate F with respect to τ

$$\frac{dF}{d\tau} = \frac{(1-\alpha_p)S}{4}\, X\, \frac{3}{4} \; or\; \Delta F = \frac{3S(1-\alpha_p)}{16}\,\Delta\tau \quad (7)$$

The following formula[2] is used to calculate $\Delta\tau$:

$$\tau_{CO_2} = 1.73\,(CO_2)^{0.263} \quad where\; CO_2\; is\; in\; ppmv\; X\; 10^{-6} \quad (8)$$

$$or\; \tau_{CO_2} = 0.457\,(CO_2)^{0.263} \quad where\; CO_2\; is\; in\; ppmv$$

Here $\tau = aC^b$ (9)

where a and b are constants

The initial conditions are $\tau_o = aC_0^b$ (10)

On dividing Equation (9) by Equation (10) and taking the natural logarithm of both sides we get:

$$\Delta\tau = \tau_0\left\{e^{b\ln\left(\frac{C}{C_0}\right)} - 1\right\} \text{ where } b = 0.263 \quad (11)$$

$$\Delta F = \frac{3S(1-\alpha_p)}{16}\,\tau_0\left\{e^{0.263\ln\left(\frac{C}{C_0}\right)} - 1\right\} \quad (12)$$

Equation 29 below can be expressed:

$$\frac{dF_{a\blacktriangleright g}}{dT} = \frac{4\sigma T^3}{A} = \frac{1}{A} X \frac{dF_{g\blacktriangleright a}}{dT} \quad where\; A = 1 + \frac{3}{4}\tau_g$$

$$\int_{T_1}^{T_2} \frac{dF_{a\blacktriangleright g}}{dT}\, dT = \frac{1}{A} X \int_{T_1}^{T_2} \frac{dF_{g\blacktriangleright a}}{dT}\, dT$$

$\Delta F_{a\blacktriangleright g} = f_a\; X\; \Delta F_{g\blacktriangleright a} \quad where\quad f_a = \frac{1}{A}\; and\; A = 1 + \frac{3}{4}\tau_g$

$f_a = 0.6$ is the fraction of flux returned downward to the Earth, absorbed or re-emitted by CO2 . This is consistent with the IPCC result. The flux density directed downward to warm the surface further is the CO_2 greenhouse flux density ($\Delta F_{a\blacktriangleright g}$) and the equation for the CO2 greenhouse flux density (W/m^2) is:

$\Delta F = \Delta F_{a \blacktriangleright g} = f_a \times \Delta F_{g \blacktriangleright a}$

Therefore, $\Delta F = f_a \frac{3S(1-\alpha_p)}{16} \tau_0 \left\{ e^{0.263 \ln\left(\frac{C}{C_0}\right)} - 1 \right\}$ (13)

We can use the following
identity to clarify the exponential term: $e^x \doteqdot 1 + x \; for \; x < 1$

$We \; set \; x = 0.263 \; \ln\left(\frac{C}{C_0}\right) \; then \; e^{0.263 \ln\left(\frac{C}{C_0}\right)} - 1 \; \doteqdot 0.263 \; \ln\left(\frac{C}{C_0}\right)$

$as \; 0.263 \; \ln\left(\frac{C}{C_0}\right) < \; 1 \; for \; C_0 = 280 \; ppm \; and \; 280 \; \leq C \; \leq 1000 \; ppm$

$\Delta F = 0.6 \times 173.0 \times 0.201 \times 0.263 \; ln \; (C/C_0)$

$\Delta F = 5.487 \; ln \; (C/C_0)$ (14)

But the IPCC equation includes a small amount of CO_2 absorption of high frequency solar radiation[6] that reduces ΔF by 0.06 W/m^2. Adjusting Equation 14 for CO_2 solar absorption gives:

$\Delta F = 5.40 \; ln \; (C/C_0)$ (15)

The coefficient, 5.40 is within the one percentage point margin of error of the IPCC result of 5.35 for their coefficient. We will use the IPCC result later.

2.2 Derivation Two

T_s is the surface air temperature and $F_{g \blacktriangleright a} = \sigma T_s^4$ is upward flux density (heat) radiated from the surface. f is the fraction of Earth's heat radiation in the spectral interval over which CO_2 absorption is significant. $f \times F_{g \blacktriangleright a}$ is the amount of Earth's heat radiation in the spectral interval over which CO_2 absorption is significant. Some of the heat radiation emanating from the surface will be absorbed by CO_2 before passing through the remainder of the atmosphere. $f \times F_{g \blacktriangleright a} \times (1 - T_d)$ is the amount of flux density from the surface absorbed by CO_2 in the upper atmosphere and re-emitted equally in all directions both upward and downward. $(1-T_d)$ is the fractional absorption. f_a is the fraction of flux density returned downward to the Earth, absorbed or re-emitted by CO_2. The flux density directed downward to warm the surface further is the CO_2 greenhouse flux density ($F_{a \blacktriangleright g}$) and the equation for the CO2 greenhouse flux density is:

$F = f_a \times f \times F_{g \blacktriangleright a} \times (1 - T_d)$
where:

F is the CO_2 greenhouse flux density in W/m^2 ($F_{a \blacktriangleright g}$)

f_a is the fraction of flux returned downward to the Earth, absorbed or re-emitted by CO_2

f is the Planck Blackbody Fraction (The fraction of Earth's heat radiation in the spectral interval over which CO_2 absorption is significant)

$F_{g\blacktriangleright a}$ is the total flux density emitted by the Earth's surface (σT_s^4)

T_d is the diffuse transmittance

$(1\text{-}T_d)$ is the fractional absorption

Opacity measures the degree of opaqueness. Infrared opacity (or optical depth) of carbon dioxide (τ_{CO_2}) measures the degree to which infrared radiation sees carbon dioxide as opaque. It describes the extent of absorption and scattering of infrared radiation and is measured downwards from the top of the atmosphere. The following formula[2] is used:

$$\tau_{CO_2} = 1.73\ (CO_2)^{0.263} \qquad where\ CO_2\ is\ in\ units\ of\ ppmv\ X\ 10^{-6}$$

The spectral interval from 550 cm^{-1} to 1015 cm^{-1} is chosen to calculate the Planck Fraction.

The formula for $1 - T_d$ expressed in terms of $\boldsymbol{\tau}$ is:

$$1 - T_d = 1 - 2\int_0^1 \mu e^{-\tau/\mu} d\mu \ = 1 - 2E_3(\tau)$$

$E_3(\tau)$ *is the exponential integral and* $E_3(0) = \frac{1}{2}$

This gives the formula: $1 - T_d = 0.05371\ \boldsymbol{ln}\ C$ where $1 \le C \le 1000$ ppmv

Using this formula we can derive the IPCC result[3] as follows:

F = 0.6 X 0.4256 X390 X 0.05371 *ln* C

$$F = 5.35\ ln\ C \qquad (16)$$

The CO2 GHG flux density (F_0) at initial concentration C_0 is given by:

$F_0 = 5.35\ ln\ C_0$ (17)

Equation 16 – Equation 17 gives:

$\Delta F = F - F_0 = 5.35\ ln\ C - 5.35\ ln\ C_0 = 5.35\ ln\ (C/C_0)$

$$\Delta F = 5.35\ \boldsymbol{ln}\ (C/C_0) \qquad (18)$$

Equation 18 is the IPCC result. The coefficient (5.35) in the IPCC result has an uncertainty of 1%. This gives the derivation from first principles of the IPCC simplified equation. We now calculate ΔF for instant doubling of the atmospheric CO_2 content by setting $C = 2C_0$ in Equation 18 and we get $\Delta F = 3.71$ W/m^2.

The logarithmic relationship between CO_2 concentration and radiative forcing (ΔF) means that each further doubling of CO_2 content gives an extra 3.71 W/m^2 and a further temperature increase. Even on Venus which has 10,000 times more CO_2 than Earth and an average surface temperature of 460^{o}C, the CO_2 absorption is not saturated. This is because on Earth, Mars and Venus it is always possible to find a higher layer of the atmosphere (with lower partial pressure, lower τ_{CO_2} and more narrow absorption lines) to absorb the heat and then radiate it up to space and down to the ground.

The equation F = 5.*35 ln* C, Equation 16 above, is important for calculating CO_2 flux density (F) at concentration C in the atmosphere. CO_2 concentration reached 400 ppm on 11th May 2013 and F is 32 W/m2 or about 20% of total greenhouse gas flux density of the Earth's Greenhouse effect. However, it is in the comparison with the *non-condensing* greenhouse gases where CO_2 gets its controlling influence. CO_2 accounts for 80% of the greenhouse gas flux density of the *non-condensing* greenhouse gases that maintain the temperature structure of the Earth and acts as the control knob of the Earth's thermostat.

5. References

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